Reformulation of Gray's duality for attractive spin systems and its applications

1994 J. Phys. A: Math. Gen. 27 3191

(http://iopscience.iop.org/0305-4470/27/9/030)

J. Phys. A: Math. Gen. 27 (1994) 3191–3211. Printed in the UK

Reformulation of Gray's duality for attractive spin systems and its applications

Makoto Katori
Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112, Japan

Received 12 October 1993

Abstract. Duality relations, which associate two Markov processes in different state spaces, have been useful tools in the study of the long-term behaviour of the stochastic processes of spin systems. In 1986, Gray introduced a new duality theory which is applicable to general spin systems with attractive transition rates. The theory was developed by making full use of graphical representations. In the present paper, Gray's duality is reformulated by studying the action of the formal generators on a newly chosen duality function and the dual processes defined in the state space $\Upsilon = \{\mathcal{A} : \mathcal{A}$ is a finite subset of $Y\}$ are discussed, where Y is a collection of finite subsets of $\mathbb{Z}^d$. As applications of the submodularity of the survival probability $\sigma(\mathcal{A})$ of the dual processes, rigorous lower bounds of the critical values are derived for the θ-contact process, the multi-particle creation model and the sexual reproduction process.

1. Introduction

There are two types of duality relations, both of which have been useful tools in the study of spin systems. One of them is the *high-temperature/low-temperature duality*, based on the duality transformations of lattices, which gives an exact evaluation of the critical temperatures for many equilibrium lattice models in statistical mechanics (see, for example, Baxter 1982). The other is the study of the stochastic processes of spin systems which associates two Markov processes in different state spaces. In the present paper, we consider the latter duality.

A general definition of this duality is the following (see section II.3 of Liggett (1985)). Suppose η_t and ζ_t are Markov processes with state spaces X and Y, respectively, and let $H(\eta, \zeta)$ be a bounded measurable function on $X \times Y$. The processes η_t and ζ_t are said to be *dual* to one another with respect to H if

$$E^{\eta}[H(\eta_t, \zeta)] = E^{\zeta}[H(\eta, \zeta_t)] \tag{1.1}$$

for all $\eta \in X$ and $\zeta \in Y$. Here the LHS represents the expectation of $H(\eta_t, \cdot)$ for the process $\eta_t \in X$ starting from the configuration η and the RHS represents that of $H(\cdot, \zeta_t)$ for ζ_t starting from ζ.

The interacting particle systems in which we are interested are usually Markov processes on an uncountable space $X = \{0, 1\}^S$ with, for example, $S = \mathbb{Z}^d$: the d-dimensional hypercubic lattice. A useful duality theory relates them to Markov chains in which the state space Y is a countable set. Sometimes the dual process is more tractable than the original process. Depending on the choice of duality function H, there are several types of duality theory.

When we analyse a spin system, we use one of these theories if we can find an appropriate simple duality function H.

In our previous paper (Katori and Konno 1993), we used a coalescing duality to study the family of one-dimensional contact processes first investigated by Durrett and Griffeath (1983), which we simply called the θ-contact process (or θ-CP for short). The θ-CP, η_t, is a one-dimensional spin system in which the state space is $X = \{0, 1\}^{\mathbb{Z}}$. The system evolves by the following single-spin-flip dynamics.

(i) If $\eta_{t^-}(x) = 1$, then $\eta_t(x) = 0$ at a rate 1.

(ii) If $\eta_{t^-}(x) = 0$, then $\eta_t(x) = 1$ at the rate $f(N_x(t^-))$ which depends on the number of the nearest-neighbour particles $N_x(t^-) = \eta_{t^-}(x-1) + \eta_{t^-}(x+1)$ as

$$f(N) = \begin{cases} 0 & \text{if } N = 0 \\ \lambda & \text{if } N = 1 \\ \theta\lambda & \text{if } N = 2. \end{cases} \tag{1.2}$$

Here λ and θ are non-negative parameters. This process can be viewed as a simple model of the spread of infection of a disease. The parameter λ is the infection rate when only one of the neighbours is infected ($N_x = 1$). The parameter θ is the ratio of the infection rate in the case $N_x = 2$ (both neighbours are infected) to that in the case $N_x = 1$. The coalescing dual process is defined on the state space

$$Y = \{A : A \text{ is a finite subset of } \mathbb{Z}\} \tag{1.3}$$

and corresponds to the choice of H as

$$H_c(\eta, A) = \prod_{x \in A} (1 - \eta(x)). \tag{1.4}$$

It is easy to show that the θ-CP has a coalescing dual process if, and only if, $1 \leqslant \theta \leqslant 2$. We extended the method of Griffeath (1975) and the Holley–Liggett argument (1978) to derive the rigorous lower and upper bounds for the critical line $\lambda = \lambda_c(\theta)$ which divides the *extinction phase* and the *survival phase*. Since all of these arguments were based on a coalescing duality, our bounds were only valid for the case $1 \leqslant \theta \leqslant 2$ (Katori and Konno 1993).

As this example shows, the coalescing duality is a powerful tool in the study of the long-term behaviour of the spin systems; however, it can only be applied to some special cases. The problem is whether we can define another duality which will cover a larger class of spin systems. In 1986, Gray introduced a new theory for dual processes which can be applied to more general spin systems. His theory is applicable to all attractive spin systems with any finite-range interactions in any dimensions (Gray 1986). Following this, we can define Gray's dual process for the θ-CP for all $\theta \geqslant 1$ (the θ-CP is attractive iff $\theta \geqslant 1$).

Gray's duality theory was developed by making full use of the graphical constructions (Gray 1986). This procedure enables us to define both the original spin system and the dual process on the same spatio-temporal hyper-plane, $\mathbb{Z}^d \times [0, t)$. On the other hand, there is another standard method for introducing a spin system by defining the corresponding Markov semigroup $S(t)$ from the formal generator Ω. The formal generator is given if we specify the flip rate $c(x, \eta)$ as a function of the spin configuration (see the next section and Liggett (1985)). Therefore, if we follow the latter procedure, we can discuss Gray's duality formally by only treating the formal generators. That is, we can reformulate Gray's duality theory by choosing a new duality function H and by observing the action of Ω on it. The state space of Gray's dual process is given by

$$\Upsilon = \{\mathcal{A} : \mathcal{A} \text{ is a finite subset of } Y\}. \tag{1.5}$$

The new choice of H is

$$H(\eta, \mathcal{A}) = \prod_{A \in \mathcal{A}} \left(1 - \prod_{x \in A} \eta(x)\right). \tag{1.6}$$

Although the state space Υ is much larger than the state space Y of the coalescing dual process, it is still countable.

In the present paper, we will explain how we can reformulate Gray's duality theory by using the formal generator Ω and (1.6) for general attractive spin systems. Almost all the results concerning this general duality were given in the original paper by Gray (1986). Our representation, however, is more similar to that for coalescing dual processes on Y and enables us to extend the methods originally used by us to Gray's dual processes. In fact, we can develop Griffeath's method (1975) to give the conditions for the extinction of processes in our framework. We define the survival probabilities $\sigma(\mathcal{A})$ for Gray's dual processes and use their generalized version of submodularity. As an application, we will show that our lower bounds for $\lambda_c(\theta)$ of θ-CP derived in the previous paper (Katori and Konno 1993) is valid not only for $1 \leqslant \theta \leqslant 2$ but also for $\theta > 2$. We will also give some lower bounds for the critical values of one-dimensional multi-particle creation models (recently studied by Dickman and Tomé (1991) and Durrett and Neuhauser (1994)), and the d-dimensional sexual reproduction processes (Noble 1992); neither of these have dual processes in the previous version.

The paper is organized as follows. In section 2, we reformulate Gray's duality by using some basic properties of H given by (1.6) when the formal generator Ω is applied. In section 3, we briefly review the duality relations and some theorems originally given by Gray (1986) in our framework. Section 4 is devoted to showing some applications of the present duality theory, which is similar to Griffeath's method (1975). Some comments are given in section 5.

2. General attractive spin systems and their dual processes

2.1. Formal generators

We consider a class of continuous-time Markov processes, η_t, on the d-dimensional hypercubic lattice $\mathbb{Z}^d$. Each site $x \in \mathbb{Z}^d$ is occupied by either a particle ($\eta_t(x) = 1$) or a vacancy ($\eta_t(x) = 0$). That is, the state space is $X = \{0, 1\}^{\mathbb{Z}^d}$. Let $C(X)$ be a set of continuous functions on X. We assume that the process follows single-spin-flip dynamics. Such an interacting particle system in which each coordinate has two possible values and only one coordinate changes in each transition is often called a *spin system*. A spin system is defined in the standard method as follows (Liggett 1985). The formal generator Ω on $C(X)$ for a spin system is given as

$$\Omega f(\eta) = \sum_{x \in \mathbb{Z}^d} c(x, \eta)[f(\eta^x) - f(\eta)] \tag{2.1}$$

for $f \in C(X)$, where

$$\eta^x(u) = \begin{cases} 1 - \eta(x) & \text{if } u = x \\ \eta(u) & \text{if } u \neq x. \end{cases} \tag{2.2}$$

If the flip rate $c(x, \eta)$ is appropriately chosen, the Markov semigroup $S(t)$ is defined by Ω as

$$S(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}\Omega \right)^{-n} f \tag{2.3}$$

for $f \in C(X)$ and $t \geqslant 0$, where I is the identity operator. There is a unique Markov process η_t corresponding to $S(t)$, such that

$$S(t)f(\eta) = E^{\eta}[f(\eta_t)] \tag{2.4}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} E^{\eta}[f(\eta_t)] = E^{\eta}[\Omega f(\eta_t)] \tag{2.5}$$

for all $f \in C(X)$, $\eta \in X$ and $t \geqslant 0$.

In this paper we assume that the flip rate is attractive: whenever $\eta \leqslant \zeta$, $c(x, \eta) \leqslant c(x, \zeta)$ if $\eta(x) = \zeta(x) = 0$ and $c(x, \eta) \geqslant c(x, \zeta)$ if $\eta(x) = \zeta(x) = 1$, and that it has a finite range: there is a finite set of sites $R_x \subset \mathbb{Z}^d$ such that $c(x, \eta)$ depends only on $X_x = \{\eta(y) : y \in R_x \cup \{x\}\}$ for each x. Such a flip rate is specified if we give the dependence of $c(x, \eta)$ on X_x as below. We call a non-empty subset in R_x a *region* and let $\mathcal{R}_x$ be a collection of these regions. We introduce an equivalence relation among the regions and this can be written as $A_x \sim B_x$ for $A_x, B_x \in \mathcal{R}_x$. We let $\tilde{\mathcal{R}}_x$ be the collection of all equivalence classes determined by this relation: $\tilde{\mathcal{R}}_x = \mathcal{R}_x / \sim$. We assume that if $\eta(x) = 0$ and if *all* sites of *at least one* of the regions, which is equivalent to $A_x \in \tilde{\mathcal{R}}_x$, are occupied by particles, then a rate $b^{(1)}_{A_x} \geqslant 0$ is added to $c(x, \eta)$. If $\eta(x) = 1$ and if *at least one* site is vacated in *all* equivalent regions to $B_x \in \tilde{\mathcal{R}}_x$, then a rate $d^{(1)}_{B_x} \geqslant 0$ is added to $c(x, \eta)$. We also assume that spin flip occurs spontaneously at a rate $b^{(1)}_{\emptyset}$, if $\eta(x) = 0$, and at a rate $d^{(1)}_{\emptyset}$ if $\eta(x) = 1$. The spin-flip rate is thus assumed to be written in the form

$$\begin{aligned} c_1(x, \eta) = (1 - \eta(x)) \left[b^{(1)}_{\emptyset} + \sum_{A_x \in \tilde{\mathcal{R}}_x} b^{(1)}_{A_x} \left\{ 1 - \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y) \right) \right\} \right] \\ + \eta(x) \left[d^{(1)}_{\emptyset} + \sum_{A_x \in \tilde{\mathcal{R}}_x} d^{(1)}_{A_x} \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y) \right) \right] \end{aligned} \tag{2.6}$$

where $b^{(1)}_{\emptyset}$, $\{b^{(1)}_{A_x}\}$, $d^{(1)}_{\emptyset}$, and $\{d^{(1)}_{A_x}\}$ are non-negative parameters. By choosing a set of these parameters appropriately, any type of attractive finite-ranged flip rates can be represented in this form.

It should be noted that when there are some simple relations among the parameters $\{b^{(1)}_{A_x}\}$ and $\{d^{(1)}_{A_x}\}$, respectively, the flip rate $c_1(x, \eta)$ given above is reduced to the following simpler form:

$$\begin{aligned} c_2(x, \eta) = (1 - \eta(x)) \left[b^{(2)}_{\emptyset} + \sum_{A_x \in \tilde{\mathcal{R}}_x} b^{(2)}_{A_x} \sum_{B_x \sim A_x} \prod_{y \in B_x} \eta(y) \right] \\ + \eta(x) \left[d^{(2)}_{\emptyset} + \sum_{A_x \in \tilde{\mathcal{R}}_x} d^{(2)}_{A_x} \sum_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y) \right) \right] \end{aligned} \tag{2.7}$$

with some non-negative parameters $b^{(2)}_{\emptyset}$, $\{b^{(2)}_{A_x}\}$, $d^{(2)}_{\emptyset}$ and $\{d^{(2)}_{A_x}\}$.

We will give some examples below.

Example 1. The θ-CP with $\theta \geqslant 1$. The θ-CP is the one-dimensional spin system with its spin-flip rate given by

$$c(x,\eta) = \lambda(1-\eta(x))\{\eta(x-1)+\eta(x+1)-(2-\theta)\eta(x-1)\eta(x+1)\}+\eta(x). \tag{2.8}$$

It is easy to confirm that this can be expressed in the form (2.6) if we choose the range and the parameters as follows. Let

$$R_x = \{x-1, x+1\} \tag{2.9}$$

and

$$\tilde{\mathcal{R}}_x = \{\{x+1\}, \{x-1, x+1\}\} \tag{2.10}$$

with $\{x-1\} \sim \{x+1\}$, and let

$$b_{A_x}^{(1)} = \begin{cases} 0 & \text{if } A_x = \emptyset \\ \lambda & \text{if } A_x = \{x+1\} \\ (\theta-1)\lambda & \text{if } A_x = \{x-1, x+1\} \end{cases} \tag{2.11}$$

and

$$d_{A_x}^{(1)} = \begin{cases} 1 & \text{if } A_x = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Example 2. The one-dimensional n-particle creation model. Dickman and Tomé (1991) studied, by using computer simulations, the one-dimensional particle systems with spin-flip rates given by the following when $n = 2$ and 3:

$$c(x,\eta) = \lambda(1-\eta(x))\left\{\prod_{i=1}^{n}\eta(x-i)+\prod_{i=1}^{n}\eta(x+i)\right\}+\eta(x). \tag{2.13}$$

This process was introduced to model autocatalytic chemical reactions. When $n = 1$ it is merely the basic contact process (i.e. $\theta = 2$ case of example 1). This flip rate is written in the form $c_2(x,\eta)$, given by (2.7), when we put

$$b_{A_x}^{(2)} = \begin{cases} \lambda & \text{if } A_x = \{x+1, x+2, \ldots, x+n\} \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

$$d_{A_x}^{(2)} = \begin{cases} 1 & \text{if } A_x = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

and assume that $\{x-n, \ldots, x-2, x-1\} \sim \{x+1, x+2, \ldots, x+n\}$.

Remark 2.1. Dickman and Tomé (1991) simulated a more general case where creation and annihilation of particles occurred with the rates (2.13) and each particle could hop to one of its neighbour sites, with a rate D, if it was vacant. The asymptotic behaviour of the process when this *diffusion* rate $D \to \infty$ was investigated by Durrett and Neuhauser (1994). Here we consider the case $D = 0$. We will comment in section 5 on the *diffusive* case.

Example 3. The d-dimensional sexual reproduction process. When $n = 2$, the above process is sometimes called the (one-dimensional) sexual reproduction process (SRP). It was studied by Noble (1992) when the diffusion rate $D \to \infty$. The d-dimensional version of the SRP is defined by the following flip rate on the lattice $\mathbb{Z}^d$ in the case $D = 0$,

$$c(x, \eta) = \lambda(1 - \eta(x)) \sum_{y:|y-x|=1} \sum_{z:|z-y|=1, z\neq x} \eta(y)\eta(z) + \eta(x). \tag{2.16}$$

It is also written in the form $c_2(x, \eta)$, given by (2.7), when we put

$$b_{A_x}^{(2)} = \begin{cases} \lambda & \text{if } A_x = \{x + e_1, x + 2e_1\} \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

$$d_{A_x}^{(2)} = \begin{cases} 1 & \text{if } A_x = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2.18}$$

and assume that $\{y, z\} \sim \{x + e_1, x + 2e_1\}$ for all y and z such that $|y - x| = 1$ and $|z - y| = 1, z \neq x$, where $e_1 \equiv (1, 0, \ldots, 0)$.

Example 4. The stochastic Ising model with ferromagnetic interaction. The spin-flip rate for the one-dimensional stochastic Ising model with nearest-neighbour interactions is usually given by (Glauber 1963, Suzuki and Kubo 1968)

$$c(x, \eta) = \tfrac{1}{2}[1 - s(x)\tanh\{h + K(s(x-1) + s(x+1))\}] \tag{2.19}$$

where $s(x) \equiv 2\eta(x) - 1 \in \{-1, 1\}$. Here $h = \beta H$ and $K = \beta J$ where H is the external field, J the exchange interaction and $\beta = 1/k_B T$ the inverse temperature It is easy to confirm that it can be written in the form $c_1(x, \eta)$, given by (2.6), when R_x and $\tilde{\mathcal{R}}_x$ are given in the same way as example 1, with $\{x - 1\} \sim \{x + 1\}$ and

$$b_{A_x}^{(1)} = \begin{cases} \frac{1}{2}(1 - \tanh(2K - h)) & \text{if } A_x = \emptyset \\ \frac{1}{2}(\tanh(2K - h) + \tanh h) & \text{if } A_x = \{x + 1\} \\ \frac{1}{2}(\tanh(2K + h) - \tanh h) & \text{if } A_x = \{x - 1, x + 1\} \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

and

$$d_{A_x}^{(1)} = \begin{cases} \frac{1}{2}(1 - \tanh(2K + h)) & \text{if } A_x = \emptyset \\ \frac{1}{2}(\tanh(2K - h) + \tanh h) & \text{if } A_x = \{x + 1\} \\ \frac{1}{2}(\tanh(2K + h) - \tanh h) & \text{if } A_x = \{x - 1, x + 1\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.21}$$

When the interaction is ferromagnetic ($J > 0$), $b_{A_x}^{(1)} \geqslant 0$ and $d_{A_x}^{(1)} \geqslant 0$: in this example, $b_{\emptyset}^{(1)} \neq 0$. The generalization for the case with long-range interactions, or for higher dimensions, is straightforward.

2.2. Notation

We will treat a collection Υ of finite subsets of Y. The elements of Υ will be denoted by $\mathcal{A}, \mathcal{B}, \ldots \in \Upsilon$. They are a collection of sets of sites, for example,

$$\begin{aligned} \mathcal{A} &= \{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_2, x_3, x_5\}\} \\ \mathcal{B} &= \{\{x_1\}, \{x_2\}, \{x_1, x_2\}\} \end{aligned} \tag{2.22}$$

where $x_i \in \mathbb{Z}^d$. An empty set in Υ is denoted by $\emptyset$ and it should be distinguished from $\{\emptyset\}$ which contains one element, an empty set of Y.

We will write the union of $\mathcal{A}$ and $\mathcal{B}$ which contains all the elements of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{A} \cup \mathcal{B}$. For example,

$$\mathcal{A} \cup \mathcal{B} = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3, x_5\}\} \tag{2.23}$$

for $\mathcal{A}$ and $\mathcal{B}$ given by (2.22). In the same way, when we write $\mathcal{A} \cap \mathcal{B}$ or $\mathcal{A} \setminus \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in \Upsilon$, they should be interpreted as the intersection and the difference in the sense of the sets in Υ (see lemma 4.1).

For a fixed site $x \in \mathbb{Z}^d$, each $\mathcal{A} \in \Upsilon$ is partitioned as

$$\mathcal{A} = \mathcal{A}(x) \cup \mathcal{A}(x)^{\mathrm{c}} \tag{2.24}$$

with

$$\mathcal{A}(x) = \{A \in \mathcal{A} : x \in A\} \qquad \mathcal{A}(x)^{\mathrm{c}} = \{A \in \mathcal{A} : x \notin A\}. \tag{2.25}$$

Then we introduce the following *operators*, a_x and $r_x(B)$, which operate on the elements of Υ:

$$a_x \mathcal{A} = \mathcal{A}(x)^{\mathrm{c}}. \tag{2.26}$$

Let $B \in Y$, then

$$r_x(B)\mathcal{A} = (r_x(B)\mathcal{A}(x)) \cup \mathcal{A}(x)^{\mathrm{c}} \tag{2.27}$$

with

$$r_x(B)\mathcal{A}(x) = \{(A \setminus \{x\}) \cup B : A \in \mathcal{A}(x)\}. \tag{2.28}$$

If $\mathcal{A}(x) = \emptyset$, we assume $r_x(B)\mathcal{A} = \mathcal{A}$. That is, a_x annihilates all sets in $\mathcal{A}$ which contain a site x and $r_x(B)$ replaces each set A in $\mathcal{A}$, which contains x, by the set which is obtained from A by removing x and adding B. We will use the following abbreviations:

$$(r_x(B_1) \cup r_x(B_2))\mathcal{A} \equiv (r_x(B_1)\mathcal{A}) \cup (r_x(B_2)\mathcal{A}) \tag{2.29}$$

and

$$\left(\bigcup_i r_x(B_i)\right)\mathcal{A} \equiv \bigcup_i (r_x(B_i)\mathcal{A}) \tag{2.30}$$

for $\mathcal{A} \in \Upsilon$, $B_i \in Y$.

2.3. Dual process

In order to introduce Gray's duality, we choose the following duality function $H(\eta, \mathcal{A})$ on $X \times \Upsilon$:

$$H(\eta, \mathcal{A}) = \begin{cases} \prod_{A \in \mathcal{A}} \left(1 - \prod_{x \in A} \eta(x)\right) & \text{if } \emptyset \notin \mathcal{A} \text{ and } \mathcal{A} \neq \emptyset \\ 0 & \text{if } \mathcal{A} \ni \emptyset \\ 1 & \text{if } \mathcal{A} = \emptyset \end{cases} \tag{2.31}$$

for $\eta \in X$.

For convenience, we introduce here some notation for combinations of operators. For $c_1(x, \eta)$ we define

$$R_1(A_x) = r_x(\{x\}) \cup \left(\bigcup_{B_x \sim A_x} r_x(B_x) \right) \tag{2.32}$$

and

$$S_1(A_x) = \bigcup_{B_x \sim A_x} r_x(\{x\} \cup B_x) \tag{2.33}$$

for each $A_x \in \tilde{\mathcal{R}}_x$. For $c_2(x, \eta)$ we define

$$R_2(A_x) = r_x(\{x\}) \cup r_x(A_x) \qquad \text{and} \qquad S_2(A_x) = r_x(\{x\} \cup A_x) \tag{2.34}$$

for each $A_x \in \mathcal{R}_x$.

We then obtain the following fundamental identities.

Proposition 2.1. Assume that the formal generators Ω_1 and Ω_2 are given by (2.1) and (2.2) with the flip rates $c_1(x, \eta)$ in the form (2.6) and $c_2(x, \eta)$ in the form (2.7), respectively. Then for any $\eta \in X, \mathcal{A} \in \Upsilon$,

$$\begin{aligned} \Omega_1 H(\eta, \mathcal{A}) = \sum_{x \in \mathbb{Z}^d : \mathcal{A}(x) \neq \emptyset} & \Big\{ b_\emptyset^{(1)} [H(\eta, r_x(\emptyset)\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + \sum_{A_x \in \tilde{\mathcal{R}}_x} b_{A_x}^{(1)} [H(\eta, R_1(A_x)\mathcal{A}) - H(\eta, \mathcal{A})] + d_\emptyset^{(1)} [H(\eta, a_x \mathcal{A}) - H(\eta, \mathcal{A})] \\ & + \sum_{A_x \in \tilde{\mathcal{R}}_x} d_{A_x}^{(1)} [H(\eta, S_1(A_x)\mathcal{A}) - H(\eta, \mathcal{A})] \Big\} \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} \Omega_2 H(\eta, \mathcal{A}) = \sum_{x \in \mathbb{Z}^d : \mathcal{A}(x) \neq \emptyset} & \Big\{ b_\emptyset^{(2)} [H(\eta, r_x(\emptyset)\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + \sum_{A_x \in \tilde{\mathcal{R}}_x} b_{A_x}^{(2)} \sum_{B_x \sim A_x} [H(\eta, R_2(B_x)\mathcal{A}) - H(\eta, \mathcal{A})] + d_\emptyset^{(2)} [H(\eta, a_x \mathcal{A}) - H(\eta, \mathcal{A})] \\ & + \sum_{A_x \in \tilde{\mathcal{R}}_x} d_{A_x}^{(2)} \sum_{B_x \sim A_x} [H(\eta, S_2(B_x)\mathcal{A}) - H(\eta, \mathcal{A})] \Big\}. \end{aligned} \tag{2.36}$$

The proof is given in appendix A.

We write the following collections of the operators by Q_i $(i = 1, 2)$ as

$$Q_1(x) = \{r_x(\emptyset), a_x\} \cup \{R_1(A_x) : A_x \in \tilde{\mathcal{R}}_x\} \cup \{S_1(A_x) : A_x \in \tilde{\mathcal{R}}_x\} \tag{2.37}$$

and

$$Q_2(x) = \{r_x(\emptyset), a_x\} \cup \{R_2(A_x) : A_x \in \mathcal{R}_x\} \cup \{S_2(A_x) : A_x \in \mathcal{R}_x\}. \tag{2.38}$$

We let $p_i(r_x(\emptyset)) = b_\emptyset^{(i)}$, $p_i(a_x) = d_\emptyset^{(i)}$ for $i = 1, 2$

$$p_1(R_1(A_x)) = b_{A_x}^{(1)} \qquad p_1(S_1(A_x)) = d_{A_x}^{(1)} \tag{2.39}$$

for $\mathcal{A}_x \in \tilde{\mathcal{R}}_x$, and

$$p_2(R_2(B_x)) = b_{A_x}^{(2)} \qquad p_2(S_2(B_x)) = d_{A_x}^{(2)} \tag{2.40}$$

for each $B_x \in \mathcal{R}_x$ such that $B_x \sim A_x$ with $A_x \in \tilde{\mathcal{R}}_x$. Then we define for $\mathcal{A}, \mathcal{B} \in \Upsilon$

$$q_i(\mathcal{A}, \mathcal{B}) = \sum_{x:\mathcal{A}(x)\neq\emptyset} \sum_{q_x \in Q_i(x): q_x\mathcal{A}=\mathcal{B}} p_i(q_x) \tag{2.41}$$

for each case $i = 1, 2$.

Then (2.35) and (2.36) can be written in the form

$$\Omega_i H(\eta, \mathcal{A}) = \sum_{\mathcal{B}} q_i(\mathcal{A}, \mathcal{B})[H(\eta, \mathcal{B}) - H(\eta, \mathcal{A})] \tag{2.42}$$

for $i = 1, 2$. Since $q_i(\mathcal{A}, \mathcal{B})$ are non-negative, they can be interpreted as the transition rates for a continuous-time Markov chain $\mathcal{A}_t$ on Υ. Since we have assumed that the range of interactions R_x is finite for any $x \in \mathbb{Z}^d$, this Markov chain is non-explosive.

We thus obtain the following main theorem.

Theorem 2.2. Let η_t be an attractive spin system with flip rates $c(x, \eta)$ given in the form (2.6) or (2.7) and let $\mathcal{A}_t$ be a Markov chain on Υ with transition rates $q(\mathcal{A}, \mathcal{B})$ given by (2.41). Then for every $\eta \in X$, $\mathcal{A} \in \Upsilon$ and $t \geqslant 0$

$$E^\eta[H(\eta_t, \mathcal{A})] = E^{\mathcal{A}}[H(\eta, \mathcal{A}_t)]. \tag{2.43}$$

It is easy to observe that the dual process $\mathcal{A}_t$ of η_t defined above with respect to (2.31) is equivalent to that defined by Gray (1986) using graphical representations.

Remark 2.2. It should be remarked here that if $\mathcal{A}_s \ni \emptyset$ at some time $s \geqslant 0$, then $\mathcal{A}_t \ni \emptyset$ for all $t \geqslant s$ since there is no such operator in $Q_i(x)$ that removes the element $\emptyset$ from $\mathcal{A}_t$. On the other hand, if $b_\emptyset^{(i)} = 0$ and $\emptyset \notin \mathcal{A}$, then $\emptyset \notin \mathcal{A}_t$ for all $t \geqslant 0$. The condition $b_\emptyset^{(i)} = 0$ is the prohibition of the spontaneous creation of particles in the original spin systems.

3. Duality relations

From now on we will assume that

$$b_{\emptyset}^{(i)} = 0. \tag{3.1}$$

Let

$$\Upsilon_0 = \{\mathcal{A} \in \Upsilon : \emptyset \notin \mathcal{A}\}. \tag{3.2}$$

As mentioned in remark 2.2, if $\mathcal{A} \in \Upsilon_0$ then $\mathcal{A}_t \in \Upsilon_0$ for all $t \geqslant 0$ under the condition (3.1). For convenience, we assume that if $\mathcal{A} = \emptyset$ then $\prod_{A \in \mathcal{A}} \cdot \equiv 1$. Then, by theorem 2.2 and the choice of (2.31) for H, we obtain the following identity for $\eta \in X, \mathcal{A} \in \Upsilon_0$ and $t \geqslant 0$

$$E^{\eta}\left[\prod_{A \in \mathcal{A}}\left(1 - \prod_{x \in A} \eta_t(x)\right)\right] = E^{\mathcal{A}}\left[\prod_{A \in \mathcal{A}_t}\left(1 - \prod_{x \in A} \eta(x)\right)\right]. \tag{3.3}$$

It is easy to see that if $\eta \equiv 1$ then

$$\prod_{A \in \mathcal{B}}\left(1 - \prod_{x \in A} \eta(x)\right) = \begin{cases} 1 & \text{if } \mathcal{B} = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Therefore, if we put $\eta \equiv 1$ in (3.3) we obtain

$$E^{1}\left[\prod_{A \in \mathcal{A}}\left(1 - \prod_{x \in A} \eta_t(x)\right)\right] = P^{\mathcal{A}}(\mathcal{A}_t = \emptyset) \tag{3.5}$$

or equivalently

$$P^{\mathcal{A}}(\mathcal{A}_t \neq \emptyset) = 1 - E^{1}\left[\prod_{A \in \mathcal{A}}\left(1 - \prod_{x \in A} \eta_t(x)\right)\right]. \tag{3.6}$$

Then we take the limit $t \to \infty$ in (3.6). Since we assume that the process is attractive there exists an upper invariant measure (Liggett 1985)

$$\mu_1 \equiv \lim_{t \to \infty} \delta_1 S(t) \tag{3.7}$$

where δ_1 is the point-mass distribution on $\eta \equiv 1$ and $S(t)$ is the Markov semigroup defined from the generator Ω by (2.3). We can thus define the *survival probability* for the dual process $\mathcal{A}_t$ by

$$\sigma(\mathcal{A}) = \lim_{t \to \infty} P^{\mathcal{A}}(\mathcal{A}_t \neq \emptyset) \tag{3.8}$$

for $\mathcal{A} \in \Upsilon_0$. The following is a corollary of theorem 2.2.

Corollary 3.1. For $\mathcal{A} \in \Upsilon_0$

$$\sigma(\mathcal{A}) = 1 - E_{\mu_1}\left[\prod_{A\in\mathcal{A}}\left(1 - \prod_{x\in A}\eta(x)\right)\right]. \tag{3.9}$$

In particular, for $A \in Y$, $A \neq \emptyset$

$$\sigma(\{A\}) = E_{\mu_1}\left[\prod_{x\in A}\eta(x)\right] = \mu_1\{\eta : \eta(x) = 1 \text{ for all } x \in A\} \tag{3.10}$$

and for $x \in \mathbb{Z}^d$

$$\sigma(\{\{x\}\}) = E_{\mu_1}[\eta(x)] = \mu_1\{\eta : \eta(x) = 1\}. \tag{3.11}$$

When $b_\emptyset^{(i)} = 0$, the lower invariant measure, $\mu_0 \equiv \lim_{t\to\infty}\delta_0 S(t)$ (δ_0 is a point-mass distribution on $\eta \equiv 0$), is δ_0. In this case it is proved that the unique stationary state is a trivial absorbing state δ_0 if, and only if, $\mu_1 = \delta_0$ (Liggett 1985). If $\mu_1 = \delta_0$, the RHS of (3.11) is zero. However, if the RHS of (3.11) is positive, then $\mu_1 \neq \delta_0$. Therefore we obtain the next theorem.

Theorem 3.2.

$$\sigma(\{\{x\}\}) = 0 \Longleftrightarrow \mu_1 = \delta_0 \tag{3.12}$$

$$\sigma(\{\{x\}\}) > 0 \Longleftrightarrow \mu_1 \neq \delta_0. \tag{3.13}$$

Remark 3.1. This theorem was given as statements (30) and (31) in Gray (1986).

Next we will provide a lemma which gives the identities between the survival probabilities. Let $u(t, \mathcal{A}) = P^{\mathcal{A}}(\mathcal{A}_t \neq \emptyset)$ for $\mathcal{A} \in \Upsilon_0$. By (2.5), (2.42) and using the duality relation (3.6) twice

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}u(t, \mathcal{A}) &= \frac{\mathrm{d}}{\mathrm{d}t}\{1 - E^1[H(\eta_t, \mathcal{A})]\} \\
&= -E^1[\Omega H(\eta_t, \mathcal{A})] \\
&= \sum_{\mathcal{B}} q_i(\mathcal{A}, \mathcal{B})[\{1 - E^1[H(\eta_t, \mathcal{B})]\} - \{1 - E^1[H(\eta_t, \mathcal{A})]\}] \\
&= \sum_{\mathcal{B}} q_i(\mathcal{A}, \mathcal{B})[u(t, \mathcal{B}) - u(t, \mathcal{A})].
\end{aligned} \tag{3.14}$$

By the definition of (3.8), we can conclude that the survival probability $\sigma(\mathcal{A})$ satisfied the following identity.

Lemma 3.3. For all $\mathcal{A} \in \Upsilon_0$

$$\sum_{\mathcal{B}} q_i(\mathcal{A}, \mathcal{B})[\sigma(\mathcal{B}) - \sigma(\mathcal{A})] = 0. \tag{3.15}$$

Remark 3.2. Before ending this section, we remark that the present duality of Gray includes the coalescing duality as a special case. Consider the case when $\mathcal{A}$ is a collection of singletons in Y

$$\mathcal{A} = \{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\} \tag{3.16}$$

with $x_i \in \mathbb{Z}^d$, $1 \leqslant i \leqslant n$. For such $\mathcal{A}$, by definition (2.31),

$$H(\eta, \mathcal{A}) = \prod_{B \in \mathcal{A}} \left(1 - \prod_{x \in B} \eta(x)\right) = \prod_{x \in A} (1 - \eta(x)) \tag{3.17}$$

when we let $A = \{x_1, x_2, \dots, x_n\}$. Let $\tilde{\Upsilon}$ be a collection of $\mathcal{A}$s which are given in the form (3.16) with some $n \geqslant 1$, i.e. the collections of singletons in Y. Then each element $\mathcal{A}$ in $\tilde{\Upsilon}$ can be identified with set $A = \{x_1, x_2, \dots, x_n\} \in Y$ and

$$H(\eta, \mathcal{A}) = H_c(\eta, A) \tag{3.18}$$

where H_c is the coalescing duality function given in (1.4). Therefore, if the time evolution of $\mathcal{A}_t$ is given such that $\mathcal{A}_t \in \tilde{\Upsilon}$ for all $t \geqslant 0$, then it is identified with the coalescing dual process A_t defined in Y.

4. Submodularity of $\sigma(\mathcal{A})$ and its applications

It is easy to prove that the survival probability $\sigma(\mathcal{A})$ satisfies the following inequality which is called *submodularity*.

Lemma 4.1. The survival probability defined by (3.8) for a dual process $\mathcal{A}_t \in \Upsilon_0$ of the spin system with $b_\emptyset = 0$ is *submodular* in the sense that

$$\sigma(\mathcal{A} \cup \mathcal{B}) + \sigma(\mathcal{A} \cap \mathcal{B}) \leqslant \sigma(\mathcal{A}) + \sigma(\mathcal{B}) \tag{4.1}$$

whenever $\mathcal{A}, \mathcal{B} \in \Upsilon_0$.

Proof. Let $h(\eta, A) \equiv 1 - \prod_{x \in A} \eta(x)$ for $\eta \in X, A \in Y$. Since $h(\eta, A) \in \{0, 1\}$ for any $\eta \in X, A \in Y$,

$$\left\{1 - \prod_{A \in \mathcal{A} \setminus \mathcal{B}} h(\eta, A)\right\} \times \prod_{B \in \mathcal{A} \cap \mathcal{B}} h(\eta, B) \times \left\{1 - \prod_{C \in \mathcal{B} \setminus \mathcal{A}} h(\eta, C)\right\} \geqslant 0 \tag{4.2}$$

for $\mathcal{A}, \mathcal{B} \in \Upsilon_0$. This is rewritten in the form

$$\prod_{A \in \mathcal{A} \cup \mathcal{B}} h(\eta, A) + \prod_{A \in \mathcal{A} \cap \mathcal{B}} h(\eta, A) \geqslant \prod_{A \in \mathcal{A}} h(\eta, A) + \prod_{A \in \mathcal{B}} h(\eta, A). \tag{4.3}$$

On the other hand, for $\mathcal{A} \in \Upsilon_0$

$$\sigma(\mathcal{A}) = 1 - E_{\mu_1}\left[\prod_{A \in \mathcal{A}} h(\eta, A)\right] \tag{4.4}$$

by the duality relation (3.9). Then (4.1) follows (4.3). □

Combining the identities given by lemma 3.3 and the inequalities between the $\sigma(\mathcal{A})$s given by lemma 4.1, we can obtain the criteria for $\sigma(\{\{x\}\}) = 0$ in some dual processes, which means the extinction of the original process $\mu_1 = \mu_0 = \delta_0$, by theorem 3.2. We will give examples below.

4.1. The θ-CP with θ ⩾ 1

We showed in section 2 that an attractive spin system, with its spin-flip rate given in the form (2.6) or (2.7), has the dual process $\mathcal{A}_t$ in Υ (whose transition rule is given by (2.42)). For the θ-CP, it is given explicitly by

$$\begin{aligned}\Omega_1 H(\eta, \mathcal{A}) = \sum_{x\in\mathbb{Z}^d:\mathcal{A}(x)\neq\emptyset} & \{\lambda[H(\eta, (r_x(\{x-1\}) \cup r_x(\{x\}) \cup r_x(\{x+1\}))\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + (\theta-1)\lambda[H(\eta, (r_x(\{x\}) \cup r_x(\{x-1, x+1\}))\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + [H(\eta, a_x\mathcal{A}) - H(\eta, \mathcal{A})]\} \qquad (4.5)\end{aligned}$$

if $\theta \geqslant 1$. Since $b_{\emptyset}^{(1)} = 0$ if $\mathcal{A} \in \Upsilon_0$ then $\mathcal{A}_t \in \Upsilon_0 {}^{\forall}t \geqslant 0$. First we explain that the present dual process will be reduced to the coalescing dual process when $1 \leqslant \theta \leqslant 2$. The following identity should be noted, which is easily confirmed by the definition of $H(\eta, \mathcal{A})$.

Lemma 4.2.

$$\begin{aligned}H(\eta, (r_x(\{x\}) \cup r_x(\{x-1, x+1\}))\mathcal{A}) &\\ = H(\eta, (r_x(\{x-1\}) \cup r_x(\{x\}))\mathcal{A}) &+ H(\eta, (r_x(\{x\}) \cup r_x(\{x+1\}))\mathcal{A}) \\ - H(\eta, (r_x(\{x-1\}) &\cup r_x(\{x\}) \cup r_x(\{x+1\}))\mathcal{A}). \qquad (4.6)\end{aligned}$$

By using (4.6), (4.5) can be rewritten as

$$\begin{aligned}\Omega_1 H(\eta, \mathcal{A}) = \sum_{x\in\mathbb{Z}^d:\mathcal{A}(x)\neq\emptyset} & \{(\theta-1)\lambda[H(\eta, (r_x(\{x-1\}) \cup r_x(\{x\}))\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + (\theta-1)\lambda[H(\eta, (r_x(\{x\}) \cup r_x(\{x+1\}))\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + (2-\theta)\lambda[H(\eta, (r_x(\{x-1\}) \cup r_x(\{x\}) \cup r_x(\{x+1\}))\mathcal{A}) - H(\eta, \mathcal{A})] \\ & + [H(\eta, a_x\mathcal{A}) - H(\eta, \mathcal{A})]\}. \qquad (4.7)\end{aligned}$$

If, and only if, $1 \leqslant \theta \leqslant 2$ and $\lambda \geqslant 0$, then $(\theta-1)\lambda$ and $(2-\theta)\lambda$ are both non-negative and can be interpreted as transition rates. In this case (4.7) shows that if $\mathcal{A} \in \tilde{\Upsilon}$ then $\mathcal{A}_t \in \tilde{\Upsilon}$ for all $t \geqslant 0$. Therefore, as mentioned in remark 3.2, the present dual process can be identified with the coalescing dual process in Y iff $1 \leqslant \theta \leqslant 2$.

Following the general theory of attractive spin systems (Liggett 1985), it can be proved that a unique critical value $\lambda_c(\theta)$ exists for each $\theta \geqslant 1$ such that

$$\begin{aligned}\lambda_c(\theta) &= \sup\{\lambda \geqslant 0 : \mu_1 = \delta_0\} \\ &= \inf\{\lambda \geqslant 0 : \mu_1 \neq \delta_0\} \qquad (4.8)\end{aligned}$$

and that when $\mu_1 = \delta_0$, all the processes should become extinct with probability one and the unique stationary state is a trivial absorbing state δ_0. Although the θ-CP is a simple spin system in one dimension, the exact value of $\lambda_c(\theta)$ has not been obtained for any value of θ. In our previous paper (Katori and Konno 1993), we proved the following lower and upper bounds of $\lambda_c(\theta)$ when $1 \leqslant \theta \leqslant 2$.

$$\lambda_L(\theta) \leqslant \lambda_c(\theta) \leqslant \lambda_U(\theta) \qquad (4.9)$$

where

$$\lambda_{\rm L}(\theta) = \frac{1}{2(\theta+1)}\left[\theta - 1 + \sqrt{\theta^2 + 10\theta + 13}\right] \tag{4.10}$$

and $\lambda_{\rm U}(\theta)$ is the largest root of the cubic equation

$$\theta\lambda^3 - (3\theta - 2)\lambda^2 - 3(2-\theta)\lambda + (2-\theta) = 0. \tag{4.11}$$

Our proof was valid only for $1 \leqslant \theta \leqslant 2$ since the argument was based on the coalescing duality theory.

Recently Jensen and Dickman (1994) studied the spin system which is equivalent to the θ-CP by the series-expansion method. Their method is not rigorous, but powerful, and gives a precise estimation of the critical values as well as the critical exponents (Dickman 1989, Jensen and Dickman 1993). They applied this series analysis to a wide range of θ for the θ-CP: $10^{-2} \leqslant \theta \leqslant 10^3$. Their estimated values exist between our lower and upper bounds when $1 \leqslant \theta \leqslant 2$. Moreover, they show that the inequalities (4.9) seem to hold not only for $1 \leqslant \theta \leqslant 2$ but also for all the values of θ they examined (Jensen and Dickman 1994).

Now we prove the following generalized version of theorem 1.2 from our previous paper (Katori and Konno 1993).

Theorem 4.3. Assume that

$$\theta \geqslant 1. \tag{4.12}$$

Then $\lambda_{\rm L}(\theta)$, defined by (4.10), gives the lower bound for the critical value

$$\lambda_{\rm L}(\theta) \leqslant \lambda_c(\theta). \tag{4.13}$$

Proof. By theorem 3.2 it is enough to prove that

$$\sigma(\{\{x\}\}) = 0 \tag{4.14}$$

when $\lambda < \lambda_{\rm L}(\theta)$. We will first prove (4.14) for $\lambda < 1$ (i) and then we will prove it for $1 \leqslant \lambda < \lambda_{\rm L}(\theta)$ (ii). Let

$$\mathcal{A}_1 = \{\{x\}\} \qquad \mathcal{A}_2 = \{\{x\}, \{x+1\}\}$$

$$\mathcal{A}_3 = \{\{x\}, \{x+1\}, \{x+2\}\} \qquad \mathcal{A}_4 = \{\{x\}, \{x-1, x+1\}\}$$

$$\mathcal{A}_5 = \{\{x\}, \{x+2\}\} \qquad \mathcal{A}_6 = \{\{x\}, \{x+1\}, \{x+2\}, \{x+3\}\}$$

$$\mathcal{A}_7 = \{\{x\}, \{x-1, x+1\}, \{x+2\}\}.$$

If we put $\mathcal{A} = \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_5$ in (3.15), we obtain the following identities by lemma 3.3.

$$\lambda\sigma(\mathcal{A}_3) + (\theta - 1)\lambda\sigma(\mathcal{A}_4) - (1 + \theta\lambda)\sigma(\mathcal{A}_1) = 0 \tag{4.15}$$

$$\sigma(\mathcal{A}_1) + \lambda\sigma(\mathcal{A}_3) - (1+\lambda)\sigma(\mathcal{A}_2) = 0 \tag{4.16}$$

$$2\sigma(\mathcal{A}_2) + \sigma(\mathcal{A}_5) + 2\lambda\sigma(\mathcal{A}_6) - (3 + 2\lambda)\sigma(\mathcal{A}_3) = 0 \tag{4.17}$$

$$\sigma(\mathcal{A}_1) + \lambda\sigma(\mathcal{A}_6) + (\theta - 1)\lambda\sigma(\mathcal{A}_7) - (1 + \theta\lambda)\sigma(\mathcal{A}_5) = 0 \tag{4.18}$$

where we have used the translation invariance and symmetry of the mechanism. On the other hand, if we let $\mathcal{A} = \mathcal{A}_1$ in (4.6) of lemma 4.2 and use the duality relation (3.9), we have

$$\sigma(\mathcal{A}_4) = 2\sigma(\mathcal{A}_2) - \sigma(\mathcal{A}_3). \tag{4.19}$$

Combining this with (4.15) and (4.16) gives the identities

$$\{\theta\lambda + (2-\theta)\}\sigma(\mathcal{A}_2) = \{\theta\lambda + (3-\theta)\}\sigma(\mathcal{A}_1) \tag{4.20}$$

and

$$\lambda\{\theta\lambda + (2-\theta)\}\sigma(\mathcal{A}_3) = \{\theta\lambda^2 + (3-\theta)\lambda + 1\}\sigma(\mathcal{A}_1). \tag{4.21}$$

The inequalities which we need are obtained by lemma 4.1 as follows. If we let $\mathcal{A} = \mathcal{A}_4$ and $\mathcal{B} = \mathcal{A}_5$ in (4.1), this gives

$$\sigma(\mathcal{A}_7) \leqslant \sigma(\mathcal{A}_4) + \sigma(\mathcal{A}_5) - \sigma(\mathcal{A}_1) \tag{4.22}$$

since $\mathcal{A}_7 = \mathcal{A}_4 \cup \mathcal{A}_5$ and $\mathcal{A}_1 = \mathcal{A}_4 \cap \mathcal{A}_5$. Similarly, we have

$$\sigma(\mathcal{A}_6) \leqslant 2\sigma(\mathcal{A}_3) - \sigma(\mathcal{A}_2) \tag{4.23}$$

and

$$\sigma(\mathcal{A}_3) \leqslant 2\sigma(\mathcal{A}_2) - \sigma(\mathcal{A}_1). \tag{4.24}$$

(i) First we apply (4.24) to (4.16). Then we obtain

$$(1-\lambda)[\sigma(\mathcal{A}_1) - \sigma(\mathcal{A}_2)] \geqslant 0 \tag{4.25}$$

which implies

$$\sigma(\mathcal{A}_1) \geqslant \sigma(\mathcal{A}_2) \tag{4.26}$$

when $\lambda < 1$. On the other hand, by duality relation (3.9),

$$\begin{aligned} \sigma(\mathcal{A}_2) &= 1 - E_{\mu_1}[(1-\eta(x))(1-\eta(x+1))] \\ &\geqslant 1 - E_{\mu_1}[1-\eta(x)] = \sigma(\mathcal{A}_1) \end{aligned} \tag{4.27}$$

since $\eta(x+1) \in \{0, 1\}$. Therefore,

$$\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2) \qquad \text{if } \lambda < 1. \tag{4.28}$$

Together with equation (4.20) this implies that

$$\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2) = 0 \qquad \text{if } \lambda < 1. \tag{4.29}$$

(ii) By using inequalities (4.22) and (4.23), we obtain from (4.15)–(4.18),

$$(\lambda^2 + 2\lambda + 2)\sigma(\mathcal{A}_1) + \lambda\sigma(\mathcal{A}_3) \geqslant (\lambda+1)^2\sigma(\mathcal{A}_5) \tag{4.30}$$

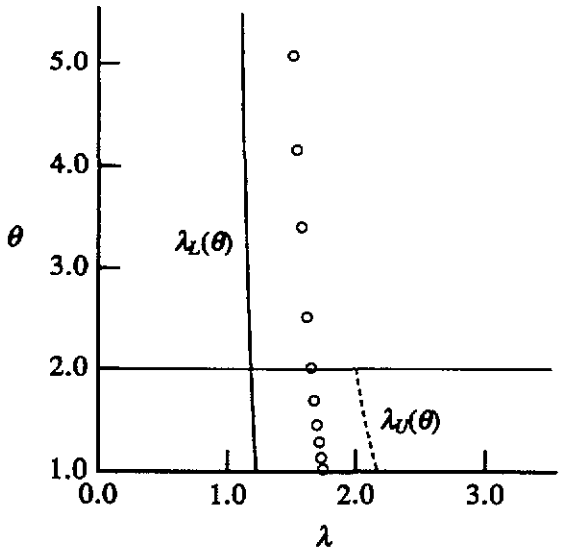

Figure 1. The full curve shows the lower bound $\lambda = \lambda_L(\theta)$ of the critical curve $\lambda = \lambda_c(\theta)$ for the θ-CP for $\theta \geqslant 1$. It is proved by theorem 4.3 that all the process should become extinct with probability one and that the unique stationary state is the trivial absorbing state δ_0 if $\lambda < \lambda_L(\theta), \theta \geqslant 1$. We show, by circles, the values of $\lambda_c(\theta)$ estimated by Jensen and Dickman (1994). The broken curve denotes the upper bound $\lambda = \lambda_U(\theta)$ which was proved only for $1 \leqslant \theta \leqslant 2$ in our previous paper (Katori and Konno 1993).

and

$$(\lambda + 1)^2\sigma(\mathcal{A}_5) \geqslant 2(\lambda^2 - 1)\sigma(\mathcal{A}_1) - (\lambda + 1)(\lambda - 3)\sigma(\mathcal{A}_3) \tag{4.31}$$

since we have assumed $\theta \geqslant 1$. Combining them gives

$$(\lambda^2 - \lambda - 3)\sigma(\mathcal{A}_3) \geqslant (\lambda^2 - 2\lambda - 4)\sigma(\mathcal{A}_1). \tag{4.32}$$

Now, we assume that $\lambda \geqslant 1$, then $\theta\lambda + (2 - \theta) \geqslant 2$. We then obtain an inequality from (4.32) and (4.21)

$$\{(\theta + 1)\lambda^2 - (\theta - 1)\lambda - 3\}\sigma(\mathcal{A}_1) \geqslant 0. \tag{4.33}$$

Since $(\theta + 1)\lambda^2 - (\theta - 1)\lambda - 3 < 0$ for $0 \leqslant \lambda < \lambda_L(\theta)$ with (4.10), it follows that

$$\sigma(\mathcal{A}_1) = 0 \qquad \text{if } 1 \leqslant \lambda < \lambda_L(\theta). \tag{4.34}$$

□

In figure 1, we show the curve $\lambda = \lambda_L(\theta)$ for $\theta \geqslant 1$ which gives the lower bound for the critical curve $\lambda = \lambda_c(\theta)$. We also plot the values of $\lambda_c(\theta)$ estimated by Jensen and Dickman (1994). As mentioned above, the validity of our lower bound has been extended for all values of $\theta \geqslant 1$ by using Gray's generalized version of duality theory. However, we have not succeeded in proving 'the upper bound' $\lambda_c(\theta) \leqslant \lambda_U(\theta)$ for $\theta > 2$, since it seems difficult to extend the Holley–Liggett argument (1978) to the dual process on Υ. So far, we have no idea how to prove such bounds of $\lambda_c(\theta)$ for the non-attractive cases $\theta < 1$.

4.2. *Multi-particle creation model and* SRP

The n-particle creation model in one dimension and the d-dimensional SRP are both attractive spin systems for any $n \geqslant 1$ and $d \geqslant 1$ and the critical values can be defined as well as the θ-CP with $\theta \geqslant 1$. We will write them as $\lambda_c^{\mathrm{MCM}}(n)$ for the multi-particle creation model and $\lambda_c^{\mathrm{SRP}}(d)$ for the d-dimensional SRP, respectively. Since $b_{\emptyset}^{(2)} = 0$ in both systems, theorem 3.2 is applicable and we can prove the following lower bounds.

Theorem 4.4.

$$\lambda_c^{\mathrm{MCM}}(n) \geqslant 1 \qquad \text{for any } n \geqslant 1 \tag{4.35}$$

$$\lambda_c^{\mathrm{SRP}}(d) \geqslant \frac{1}{d(2d-1)} \qquad \text{for any } d \geqslant 1. \tag{4.36}$$

These bounds can be proved by using lemmas 3.3 and 4.1 in the same way as the θ-CP. More detail is given in appendix B. This result is a simple generalization of the bound (1.5) in Noble (1992).

5. Comments

In the present paper, we reformulate Gray's duality theory for attractive spin systems with spin-flip rates given in the form (2.6) or (2.7). The argument can be extended to the system where each particle can hop to a vacant site with some rate. Such a process is called *the exclusion process* and its formal generator is given by

$$\Omega_e f(\eta) = \sum_{\eta(x)=1} \sum_{\eta(y)=0} h(x,y)[f(\eta^{xy}) - f(\eta)] \tag{5.1}$$

where

$$\eta^{xy}(u) = \begin{cases} \eta(y) & \text{if } u = x \\ \eta(x) & \text{if } u = y \\ \eta(u) & \text{otherwise.} \end{cases} \tag{5.2}$$

Here $h(x, y)$ denotes the hopping rate from x to y and we will assume that it is symmetric

$$h(x,y) = h(y,x). \tag{5.3}$$

For example, when we consider the nearest-neighbour hopping with a constant rate D, we let

$$h(x,y) = \begin{cases} D & \text{if } |x-y| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

For $\mathcal{A} \in \Upsilon_0$ we define $\mathcal{A}(x, y^c) = \{A \in \mathcal{A} : x \in A \text{ and } y \notin A\}$ for $x, y \in \mathbb{Z}^d$. Then we have

$$\Omega_e H(\eta, \mathcal{A}) = \sum_{x:\mathcal{A}(x)\neq\emptyset} \sum_{y:\mathcal{A}(x,y^c)\neq\emptyset} h(x,y)[H(\eta, (r_x(\{y\})r_y(\{x\}))\mathcal{A}) - H(\eta, \mathcal{A})]. \tag{5.5}$$

In this paper, we have defined the survival probability $\sigma(\mathcal{A})$ for $\mathcal{A} \in \Upsilon_0$ and have shown that it is submodular. We have applied the method of Griffeath (1975) which gave the criterion for the extinction $\mu_1 = \delta_0$ of the original spin system by using submodularity. Although his method was originally used for coalescing processes, we have shown here that it also works well in Gray's version and have given lower bounds to the critical values for the θ-CP, the multi-particle creation model and the SRP. The coalescing duality theory has been useful for discussing not only the extinction ($\mu_1 = \delta_0$) but also the survival of the process ($\mu_1 \neq \delta_0$). A typical example is the Holley–Liggett argument for contact processes (Holley and Liggett 1978, Liggett 1991a,b, Katori and Konno 1993). The basic properties of the processes defined in the state space Υ should be studied further.

Acknowledgments

I wish to thank I Jensen and R Dickman for sending their paper prior to publication. I also thank N Konno for valuable comments. The present work is partially financed by the Institute of Science and Engineering, Chuo University, Japan.

Appendix A. Proof of proposition 2.1

By definition (2.31) of $H(\eta, \mathcal{A})$ we have the following lemma.

Lemma A.1. For $\mathcal{A}, \mathcal{B} \in \Upsilon$ and $\eta \in X$

$$H(\eta, \mathcal{A} \cup \mathcal{B}) = H(\eta, \mathcal{A}) H(\eta, \mathcal{B}). \tag{A.1}$$

In particular,

$$H(\eta, \mathcal{A}) = H(\eta, \mathcal{A}(x)) H(\eta, \mathcal{A}(x)^{c}) \tag{A.2}$$

for $\mathcal{A} \in \Upsilon, x \in \mathbb{Z}^d$, where $\mathcal{A}(x)$ and $\mathcal{A}(x)^c$ are defined by (2.24) and (2.25).

Here we introduce an operator $c(B)$ with $B \in Y$ operating on the elements in Υ such that

$$c(B)\mathcal{A} = \{A \cup B : A \in \mathcal{A}\}. \tag{A.3}$$

It is noted that

$$c(B) r_x(\emptyset) \mathcal{A}(x) = r_x(B) \mathcal{A}(x) \tag{A.4}$$

and

$$c(B) \mathcal{A}(x) = r_x(\{x\} \cup B) \mathcal{A}(x) \tag{A.5}$$

if $\mathcal{A}(x) \neq \emptyset$, where $r_x(B)$ was defined by (2.27) and (2.28).

It is easy to confirm the following identities.

Lemma A.2.

(1) Let $A \in Y, \mathcal{B} \in \Upsilon$ and $\eta \in X$. If $A \neq \emptyset$ and $\mathcal{B} \neq \emptyset$, then

$$\begin{aligned} H(\eta, c(A)\mathcal{B}) &= \prod_{x\in A} \eta(x) H(\eta, \mathcal{B}) + \left(1 - \prod_{x\in A} \eta(x)\right) \\ &= H(\eta, \mathcal{B}) + \left(1 - \prod_{x\in A} \eta(x)\right)(1 - H(\eta, \mathcal{B})). \end{aligned} \tag{A.6}$$

(2) Assume that $\mathcal{A}(x) \neq \emptyset$, then

$$\eta(x)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))] = 1 - H(\eta, \mathcal{A}(x)) \tag{A.7}$$

and

$$(2\eta(x) - 1)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))] = 1 - H(\eta, r_x(\emptyset)\mathcal{A}(x)) \tag{A.8}$$

where $\eta \in X$ and η^x is defined as (2.2).

Combining lemmas A.1 and A.2 gives the following identities.

Lemma A.3. Assume that $\mathcal{A}(x) \neq \emptyset$.

$$(1 - \eta(x))[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))] = H(\eta, r_x(\emptyset)\mathcal{A}(x)) - H(\eta, \mathcal{A}(x)) \tag{A.9}$$

$$\begin{aligned} &(1 - \eta(x))\left\{1 - \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)\right\}[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))] \\ &\quad = H\left(\eta, \left(r_x(\{x\}) \cup \left(\bigcup_{B_x \sim A_x} r_x(B_x)\right)\right)\mathcal{A}(x)\right) - H(\eta, \mathcal{A}(x)) \end{aligned} \tag{A.10}$$

$$\begin{aligned} &\eta(x) \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))] \\ &\quad = H\left(\eta, \left(\bigcup_{B_x \sim A_x} r_x(\{x\} \cup B_x)\right)\mathcal{A}(x)\right) - H(\eta, \mathcal{A}(x)) \end{aligned} \tag{A.11}$$

$$\begin{aligned} &(1 - \eta(x)) \prod_{y \in B_x} \eta(y)[H(\eta^x, \mathcal{A}(x)) - H(\eta. \mathcal{A}(x))] \\ &\quad = H(\eta, (r_x(\{x\}) \cup r_x(B_x))\mathcal{A}(x)) - H(\eta, \mathcal{A}(x)). \end{aligned} \tag{A.12}$$

Proof. Here we give the proof for (A.10). The other identities can be proved in the same way. By (A.1) and (A.6), for $\mathcal{B} \in \Upsilon$,

$$\begin{aligned} H\left(\eta, \left(c(\{x\}) \cup \left(\bigcup_{B_x \sim A_x} c(B_x)\right)\right)\mathcal{B}\right) &= H(\eta, c(\{x\})\mathcal{B}) \times \prod_{B_x \sim A_x} H(\eta, c(B_x)\mathcal{B}) \\ &= H(\eta, \mathcal{B}) + (1 - \eta(x)) \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)(1 - H(\eta, \mathcal{B})). \end{aligned} \tag{A.13}$$

Let $\mathcal{B} = r_x(\emptyset)\mathcal{A}(x)$ in (A.13) and use (A.4), (A.7) and (A.8), We then have

$$H\left(\eta, \left(r_x(\{x\}) \cup \left(\bigcup_{B_x \sim A_x} r_x(B_x)\right)\right)\mathcal{A}(x)\right) = 1 - (2\eta(x) - 1)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))]$$

$$- (1 - \eta(x)) \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))]$$

$$= 1 - \eta(x)[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))]$$

$$+ (1 - \eta(x))\left\{1 - \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)\right\}[H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))]$$

$$= H(\eta, \mathcal{A}(x)) + (1 - \eta(x))\left\{1 - \prod_{B_x \sim A_x} \left(1 - \prod_{y \in B_x} \eta(y)\right)\right\}$$

$$\times [H(\eta^x, \mathcal{A}(x)) - H(\eta, \mathcal{A}(x))]. \quad \text{(A.14)}$$

□

Proposition 2.1 follows lemmas A.1, A.2 and A.3.

Appendix B. Proof of theorem 4.4

For the one-dimensional n-particle creation model, the survival probabilities of the dual process should satisfy

$$\sum_{x \in \mathbb{Z}: \mathcal{A}(x) \neq \emptyset} \{[\sigma(a_x \mathcal{A}) - \sigma(\mathcal{A})] + \lambda[\sigma(\mathcal{A} \cup r_x(\{x+1, x+2, \ldots, x+n\})\mathcal{A}) - \sigma(\mathcal{A})]$$

$$+ \lambda[\sigma(\mathcal{A} \cup r_x(\{x-n, \ldots, x-2, x-1\})\mathcal{A}) - \sigma(\mathcal{A})]\} = 0 \quad \text{(B.1)}$$

for $\mathcal{A} \in \Upsilon_0$ by lemma 3.3. Let

$$A_1 = \{x+1, x+2, \ldots, x+n\}$$

$$A_2 = \{x-n, \ldots, x-2, x-1\}$$

$$A_3 = \{x+2, x+3, \ldots, x+n+1\}$$

$$A_4^{(k)} = \{x+1, \ldots, x+k-1, x+k+1, \ldots, x+k+n\}$$

for $2 \leqslant k \leqslant n$. Then by letting $\mathcal{A} = \{\{x\}\}, \{\{x\}, A_1\}$ and $\{A_1\}$ in (B.1), we have

$$2\lambda\sigma(\{\{x\}, A_1\}) - (2\lambda + 1)\sigma(\{\{x\}\}) = 0 \quad \text{(B.2)}$$

$$\sigma(\{A_1\}) + n\sigma(\{\{x\}\}) + \lambda\sigma(\{A_2, \{x\}, A_1\}) + \lambda\sigma(\{\{x\}, A_1, A_3\})$$

$$+ \lambda \sum_{k=2}^{n} \sigma(\{\{x\}, A_1, A_4^{(k)}\}) - (n+1)(\lambda+1)\sigma(\{\{x\}, A_1\}) = 0 \quad \text{(B.3)}$$

and

$$\lambda\sigma(\{A_1, A_3\}) + \lambda \sum_{k=2}^{n} \sigma(\{A_1, A_4^{(k)}\}) - \frac{n}{2}(2\lambda + 1)\sigma(\{A_1\}) = 0. \tag{B.4}$$

By the submodularity of lemma 4.1 and the translation invariance and symmetry of the mechanism, we have

$$\sigma(\{A_2, \{x\}, A_1\}) \leqslant 2\sigma(\{\{x\}, A_1\}) - \sigma(\{\{x\}\}) \tag{B.5}$$

$$\sigma(\{\{x\}, A_1, A_3\}) \leqslant \sigma(\{\{x\}, A_1\}) + \sigma(\{A_1, A_3\}) - \sigma(\{A_1\}) \tag{B.6}$$

$$\sigma(\{\{x\}, A_1, A_4^{(k)}\}) \leqslant \sigma(\{\{x\}, A_1\}) + \sigma(\{A_1, A_4^{(k)}\}) - \sigma(\{A_1\}). \tag{B.7}$$

Applying these inequalities to (B.3) and using (B.4), we obtain

$$\{\lambda - (n+1)\}\sigma(\{\{x\}, A_1\}) + \frac{1}{2}(n+2)\sigma(\{A_1\}) - (\lambda - n)\sigma(\{\{x\}\}) \geqslant 0 \tag{B.8}$$

which together with (B.2) implies

$$(n+2)\lambda\sigma(\{A_1\}) - \{\lambda + (n+1)\}\sigma(\{\{x\}\}) \geqslant 0. \tag{B.9}$$

On the other hand, by (3.10) of corollary 3.1

$$\begin{aligned}\sigma(\{A_1\}) &= E_{\mu_1}\left[\prod_{i=1}^{n} \eta(x+i)\right] \\ &\leqslant E_{\mu_1}[\eta(x)] = \sigma(\{\{x\}\})\end{aligned} \tag{B.10}$$

since $\eta(x+i) \in \{0, 1\}$ and the system is translation invariant. Therefore

$$(n+1)(\lambda - 1)\sigma(\{\{x\}\}) \geqslant 0 \tag{B.11}$$

which means that

$$\sigma(\{\{x\}\}) = 0 \qquad \text{if } \lambda < 1.$$

By theorem 3.2, this implies $\lambda_c^{\mathrm{MCM}}(n) \geqslant 1$.

The proof for $\lambda_c^{\mathrm{SRP}}(d) \geqslant 1/d(2d-1)$ is given in the same way. □

References

Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)

Dickman R 1989 Non-equilibrium lattice model: series analysis of steady states *J. Stat. Phys.* **55** 997–1026

Dickman R and Tomé T 1991 First-order phase transition in a one-dimensional non-equilibrium model *Phys. Rev.* A **44** 4833–8

Durrett R and Griffeath D 1983 Supercritical contact processes on Z *Ann. Probab.* **11** 1–15

Durrett R and Neuhauser C 1994 Particle systems and reaction-diffusion equations *Ann. Prob.* **22** 289–333

Glauber R J 1963 Time-dependent statistics of the Ising model *J. Math. Phys.* **4** 294–307

Gray L 1986 Duality for general attractive spin systems with applications in one dimension *Ann. Prob.* **14** 371–96

Griffeath D 1975 Ergodic theorems for graph interactions *Adv. Appl. Prob.* **7** 179–94

Holley R and Liggett T M 1978 The survival of contact processes *Ann. Prob.* **6** 198–206

Jensen I and Dickman R 1993 Time-dependent perturbation theory for non-equilibrium lattice models *J. Stat. Phys.* **71** 89–127

—— 1994 Series analysis of the generalized contact process *Physica* **203A** 175–88

Katori M and Konno N 1993 Bounds for the critical line of the θ-contact processes with $1 \leqslant \theta \leqslant 2$ *J. Phys. A: Math. Gen.* **26** 6597–614

Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)

—— 1991a Spatially inhomogeneous contact processes *Spatial Stochastic Processes* ed K S Alexander and J C Watkins (Boston: Birkhäuser) pp 105–40

—— 1991b The periodic threshold contact process *Random Walks, Brownian Motion and Interacting Particle Systems* ed R Durrett and H Kesten (Boston: Birkhäuser) pp 339–58

Noble C 1992 Equilibrium behaviour of the sexual reproduction process with rapid diffusion *Ann. Prob.* **20** 724–45

Suzuki M and Kubo R 1968 Dynamics of the Ising model near the critical point. I *J. Phys. Soc. Japan* **24** 51–60